

REVERSES AND VARIATIONS OF HEINZ INEQUALITY

MOJTABA BAKHERAD¹ AND MOHAMMAD SAL MOSLEHIAN²

ABSTRACT. Let A, B be positive definite $n \times n$ matrices. We present several reverse Heinz type inequalities, in particular

$$\|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 \leq \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2,$$

where X is an arbitrary $n \times n$ matrix, $\|\cdot\|_2$ is Hilbert-Schmidt norm and $\nu > 1$. We also establish a Heinz type inequality involving the Hadamard product of the form

$$2\|A^{\frac{1}{2}} \circ B^{\frac{1}{2}}\| \leq \|A^s \circ B^{1-t} + A^{1-s} \circ B^t\| \leq \max\{\|(A+B) \circ I\|, \|(A \circ B) + I\|\},$$

in which $s, t \in [0, 1]$ and $\|\cdot\|$ is a unitarily invariant norm.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field \mathbb{C} . An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive (positive semidefinite for matrices) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. The set of all positive invertible operators (respectively, positive definite matrices) is denoted by $\mathbb{B}(\mathcal{H})_{++}$ (respectively, \mathcal{P}_n).

The Gelfand map $f(t) \mapsto f(A)$ is an isometrically $*$ -isomorphism between the C^* -algebra $C(\sigma(A))$ of all continuous functions on the spectrum $\sigma(A)$ of a selfadjoint operator A and the C^* -algebra generated by A and the identity operator I such that If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ ($t \in \sigma(A)$) implies that $f(A) \geq g(A)$.

If $\{e_j\}$ is an orthonormal basis of \mathcal{H} , $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is the isometry defined by $Ve_j = e_j \otimes e_j$ and $A \otimes B$ is the tensor product of operators A, B , then the Hadamard product $A \circ B$ regarding to $\{e_j\}$ is expressed by $A \circ B = V^*(A \otimes B)V$.

A unitarily invariant norm $\|\cdot\|$ is defined on a norm ideal $\mathfrak{L}_{\|\cdot\|}$ of $\mathbb{B}(\mathcal{H})$ associated with it and has the property $\|UXV\| = \|X\|$, where U and V are arbitrary unitaries in $\mathbb{B}(\mathcal{H})$ and $X \in \mathfrak{L}_{\|\cdot\|}$. A compact operator $A \in \mathbb{B}(\mathcal{H})$ is called Hilbert-Schmidt if $\|A\|_2 = \left(\sum_{j=1}^{\infty} s_j^2(A)\right)^{1/2} < \infty$, where $s_1(A), s_2(A), \dots$ are the singular values of A , i.e.,

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the eigenvalues of the positive operator $|A| = (A^*A)^{\frac{1}{2}}$ enumerated as $s_1(A) \geq s_2(A) \geq \dots$ with their multiplicities counted. The Hilbert-Schmidt norm is a unitarily invariant norm. For $A = [a_{ij}] \in \mathbb{M}_n$, it holds that $\|A\|_2 = \left(\sum_{i,j=1}^n |a_{i,j}|^2 \right)^{1/2}$. For two operators $A, B \in \mathbb{B}(\mathcal{H})_{++}$, let $A \sharp_{\mu} B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\mu} A^{\frac{1}{2}}$ ($\mu \in \mathbb{R}$). The operators $A \sharp_{\frac{1}{2}} B$ and $A \nabla B = \frac{A+B}{2}$ are called the operator geometric mean and the operator arithmetic mean, respectively.

The Heinz mean is defined by

$$H_{\nu}(a, b) = \frac{a^{\nu} b^{1-\nu} + a^{1-\nu} b^{\nu}}{2} \quad (0 \leq \nu \leq 1, a, b > 0).$$

The function H_{ν} is symmetric about the point $\nu = \frac{1}{2}$. Note that $H_0(a, b) = H_1(a, b) = \frac{a+b}{2}$, $H_{1/2}(a, b) = \sqrt{ab}$ and $H_{1/2}(a, b) \leq H_{\nu}(a, b) \leq H_0(a, b)$ for all $\nu \in [0, 1]$.

The Heinz norm (double) inequality, which is one of the essential inequalities in operator theory, states that for any positive operators $A, B \in \mathbb{B}(\mathcal{H})$, any operator $X \in \mathbb{B}(\mathcal{H})$ and any $\nu \in [0, 1]$, the double inequality

$$2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leq \|AX + XB\| \quad (1.1)$$

holds; see [7]. Bhatia and Davis [3] proved that (1.1) is valid for any unitarily invariant norm. Fujii et al. [6] proved that the right hand side inequality at (1.1) is equivalent to several other norm inequalities such as

- (i) the McIntosh inequality [13] asserting that $\|A^*AX + XB^*B\| \geq 2\|AXB^*\|$ for all $A, B, X \in \mathbb{B}(\mathcal{H})$;
- (ii) the Corach–Porta–Recht inequality $\|AXA^{-1} + A^{-1}XA\| \geq 2\|X\|$, where $A \in \mathbb{B}(\mathcal{H})$ is selfadjoint and invertible and $X \in \mathbb{B}(\mathcal{H})$ (see also [4]), and
- (iii) the inequality $\|A^{2m+n}XB^{-n} + A^{-n}XB^{2m+n}\| \geq \|A^{2m}X + XB^{2m}\|$ in which A, B are invertible self-adjoint operators, X is an arbitrary operator in $\mathbb{B}(\mathcal{H})$ and both m and n are nonnegative integers; see also Section 3.9 of the monograph [5].

Audenaert [1] gave a singular value inequality for the Heinz means of matrices as follows: If $A, B \in \mathbb{M}_n$ are positive semidefinite and $\nu \in [0, 1]$, then

$$s_j(A^{\nu}B^{1-\nu} + A^{1-\nu}B^{\nu}) \leq s_j(A + B).$$

Kittaneh and Manasrah [10] showed a refinement of the right hand side of inequality (1.1) for the Hilbert-Schmidt norm as follows:

$$\|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\|_2^2 + 2r_0\|AX - XB\|_2^2 \leq \|AX + XB\|_2^2, \quad (1.2)$$

in which $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite, $\nu \in [0, 1]$ and $r_0 = \min\{\nu, 1-\nu\}$. Kaur et al. [8], by using the convexity of the function $f(\nu) = \|A^{1-\nu}XB^{\nu} +$

$A^\nu XB^{1-\nu}$ ($\nu \in [0, 1]$) presented more refinements of the Heinz inequality. More precisely, for $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite and $\nu \in [0, 1]$, they showed the inequality

$$|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leq |||4r_1A^{\frac{1}{2}}XB^{\frac{1}{2}} + (1 - 2r_1)(AX + XB)|||,$$

where $r_1 = \min \left\{ \nu, \left| \frac{1}{2} - \nu \right|, 1 - \nu \right\}$. It is shown in [11] a reverse of inequality (1.2) as

$$\|AX + XB\|_2^2 \leq \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2 + 2r_0\|AX - XB\|_2^2, \quad (1.3)$$

where $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite, $\nu \in [0, 1]$ and $r_0 = \max\{\nu, 1 - \nu\}$. Aujla [16] showed that

$$2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq |||A^sXB^{1-t} + A^{1-s}XB^t|||,$$

where $A, B, X \in \mathbb{M}_n$ such that A, B are positive semidefinite, $s, t \in [0, 1]$. It is remarkable that, by using the fact that the function $g(s, t) = |||A^sXB^{1-t} + A^{1-s}XB^t|||$ attains its maximum at the vertices of the square $[0, 1] \times [0, 1]$, one can see that under the same conditions as above

$$|||A^sXB^{1-t} + A^{1-s}XB^t||| \leq \max \{ |||AX + XB|||, |||AXB + X||| \},$$

Recently, Krnić et al. used the Jensen functional to improve several Heinz type inequalities [12].

In this paper, we obtain a reverse of (1.2) and some other operator inequalities. We also show some results on the Hadamard product. In particular, we get the following Heinz type inequality

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^s \circ B^{1-t} + A^{1-s} \circ B^t||| \leq \max \{ |||(A + B) \circ I|||, |||(A \circ B) + I||| \},$$

where $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $s, t \in [0, 1]$.

2. A REVERSE OF THE HEINZ INEQUALITY FOR MATRICES

In this section, we present a converse of the Heinz inequality and give several refinements for matrices.

Lemma 2.1. *Let $a, b > 0$ and $\nu \notin [0, 1]$. Then*

$$a + b \leq a^\nu b^{1-\nu} + b^\nu a^{1-\nu}. \quad (2.1)$$

Proof. Let $\nu \notin [0, 1]$. Assume that $f(t) = t^{1-\nu} - \nu + (\nu - 1)t$ ($t \in (0, \infty)$). It is easy to see that $f(t)$ has a minimum at $t = 1$ in the interval $(0, \infty)$. Hence $f(t) \geq f(1) = 0$ for all $t > 0$. Assume that $a, b > 0$. Letting $t = \frac{b}{a}$, we get

$$\nu a + (1 - \nu)b \leq a^\nu b^{1-\nu}. \quad (2.2)$$

Applying (2.2) we obtain

$$\nu a + (1 - \nu)b \leq a^\nu b^{1-\nu} \quad \text{and} \quad \nu b + (1 - \nu)a \leq b^\nu a^{1-\nu},$$

whence

$$a + b \leq a^\nu b^{1-\nu} + b^\nu a^{1-\nu}.$$

□

For $\nu \notin [0, 1]$, if we replace ν by $\nu/(2\nu - 1)$ and A, B, X by $A^{2\nu-1}, B^{2\nu-1}, A^{1-\nu}XB^{1-\nu}$ in (1.1), respectively, then we reach the following Theorem, complementary to the right inequality in (1.1).

Theorem 2.2. *Let $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $\nu \notin [0, 1]$. Then*

$$|||AX + XB||| \leq |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||.$$

In the next theorem we show a reverse of (1.2). First, we need the following lemma.

Lemma 2.3. *Let $a, b > 0$ and $\nu \notin [\frac{1}{2}, 1]$. Then*

- (i) $\nu a + (1 - \nu)b + (\nu - 1)(\sqrt{a} - \sqrt{b})^2 \leq a^\nu b^{1-\nu}$
- (ii) $(a + b) + 2(\nu - 1)(\sqrt{a} - \sqrt{b})^2 \leq a^\nu b^{1-\nu} + b^\nu a^{1-\nu}$
- (iii) $(a + b)^2 + 2(\nu - 1)(a - b)^2 \leq (a^\nu b^{1-\nu} + b^\nu a^{1-\nu})^2$.

Proof. Let $a, b > 0$ and $\nu \notin [\frac{1}{2}, 1]$.

(i) By inequality (2.2),

$$\begin{aligned} \nu a + (1 - \nu)b + (\nu - 1)(\sqrt{a} - \sqrt{b})^2 &= (2 - 2\nu)\sqrt{ab} + (2\nu - 1)a \\ &\leq (\sqrt{ab})^{2-2\nu} a^{2\nu-1} = a^\nu b^{1-\nu}. \end{aligned}$$

(ii) It can be proved in a similar fashion as (i).

(iii) It follows from (ii) by replacing a by a^2 and b by b^2 . □

Theorem 2.4. *Suppose that $A, B \in \mathcal{P}_n$, $X \in \mathbb{M}_n$ and $\nu > 1$. Then*

$$\|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 \leq \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2.$$

Proof. By the spectral decomposition [17, Theorem 3.4], there are unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U\Lambda U^*$ and $B = V\Gamma V^*$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$, and λ_j, γ_j ($j = 1, \dots, n$) are eigenvalues of A, B , respectively. These numbers are positive. If $Z = U^*XV = [z_{ij}]$, then

$$AX + XB = U(\Lambda Z + Z\Gamma)V^* = U\left[\left(\lambda_i + \gamma_j\right)z_{ij}\right]V^*, \quad (2.3)$$

$$AX - XB = U\Lambda U^*X - XV\Gamma V^* = U[\Lambda Z - Z\Gamma]V^* = U\left[\left(\lambda_i - \gamma_j\right)z_{ij}\right]V^* \quad (2.4)$$

and

$$\begin{aligned} A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu &= U\Lambda^\nu U^*XV\Gamma^{1-\nu}V^* + U\Lambda^{1-\nu}U^*XV\Gamma^\nu V^* \\ &= U\Lambda^\nu Z\Gamma^{1-\nu}V^* + U\Lambda^{1-\nu}Z\Gamma^\nu V^* \\ &= U\left[\Lambda^\nu Z\Gamma^{1-\nu} + \Lambda^{1-\nu}Z\Gamma^\nu\right]V^* \\ &= U\left[\left(\lambda_i^\nu \gamma_j^{1-\nu} + \lambda_i^{1-\nu} \gamma_j^\nu\right)z_{ij}\right]V^*. \end{aligned} \quad (2.5)$$

It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned} \|AX + XB\|_2^2 + 2(\nu - 1)\|AX - XB\|_2^2 &= \sum_{i,j=1}^n \left(\lambda_i + \gamma_j\right)^2 |z_{ij}|^2 + 2(\nu - 1) \sum_{i,j=1}^n \left(\lambda_i - \gamma_j\right)^2 |z_{ij}|^2 \\ &\leq \sum_{i,j=1}^n \left(\lambda_i^\nu \gamma_j^{1-\nu} + \lambda_i^{1-\nu} \gamma_j^\nu\right)^2 |z_{ij}|^2 \quad (\text{by Lemma 2.3 (iii)}) \\ &= \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2^2. \end{aligned}$$

□

Remark 2.5. Utilizing Lemma 2.3, one can easily see that Theorem 2.4 holds for $\nu < \frac{1}{2}$. The case $\nu < \frac{1}{2}$ is not interesting since the left hand side is less precise than the left hand side of Theorem 2.2, but the case of $0 \leq \nu \leq \frac{1}{2}$ coincides with inequality (1.3).

Theorem 2.4 yields the next two corollaries.

Corollary 2.6. *Suppose that $A, B \in \mathcal{P}_n, X \in \mathbb{M}_n$ and $\nu > 1$. Then*

$$\|AX + XB\|_2 = \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2$$

if and only if $AX = XB$.

Proof. If $AX = XB$, then $A^\nu X = XB^\nu$ and $A^{1-\nu}X = XB^{1-\nu}$. Hence

$$\|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2 = \|A^\nu A^{1-\nu}X + XB^{1-\nu}B^\nu\|_2 = \|AX + XB\|_2.$$

Conversely, assume that $\|AX + XB\|_2 = \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2$. It follows from Theorem 2.4 that $\|AX - XB\|_2 = 0$. Thus $AX = XB$. \square

Corollary 2.7. *Let $A, B \in \mathcal{P}_n$ and $\nu > 1$. Then*

$$s_j(A + B) = s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \quad (j = 1, 2, \dots, n)$$

if and only if $A = B$.

Proof. If $A = B$, then $A + B = A^\nu B^{1-\nu} + A^{1-\nu}B^\nu$. Conversely, assume that $s_j(A + B) = s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu)$ ($j = 1, 2, \dots, n$). Then $\|AX + XB\|_2 = \|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\|_2$. It follows from Corollary 2.6 that $A = B$. \square

3. A REVERSE OF THE HEINZ INEQUALITY FOR OPERATORS

In this section we obtain a reverse of the Heinz inequality for two positive invertible operators as well as some other operator inequalities.

In [9], the authors investigated an operator version of the classical Heinz mean, i.e., the operator

$$H_\nu(A, B) = \frac{A \sharp_\nu B + A \sharp_{1-\nu} B}{2}, \quad (3.1)$$

where $A, B \in \mathbb{B}(\mathcal{H})_{++}$, and $\nu \in [0, 1]$. As in the real case, this mean interpolates between arithmetic and geometric mean, that is,

$$A \sharp B \leq H_\nu(A, B) \leq A \nabla B.$$

On the other hand, since $A, B \in \mathbb{B}(\mathcal{H})_{++}$, the expression (3.1) is also well-defined for $\nu \notin [0, 1]$. Using inequality (2.2) and the functional calculus for $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ we get the following result.

$$H_{1-\nu}(A, B) = \frac{A \sharp_{1-\nu} B + A \sharp_\nu B}{2} \geq \frac{A \nabla_{1-\nu} B + A \nabla_\nu B}{2} = A \nabla B, \quad (3.2)$$

where $A, B \in \mathbb{B}(\mathcal{H})_{++}$ and $\nu \notin [0, 1]$. Applying Lemma 2.3 (ii), we have a refinement of inequality (3.2).

Theorem 3.1. *Let $A, B \in \mathbb{B}(\mathcal{H})_{++}$ and $\nu > 1$. Then*

$$A \nabla B + 2(\nu - 1)(A \nabla B - A \sharp_{1/2} B) \leq H_{1-\nu}(A, B).$$

Proof. By Lemma 2.3 (ii), we have $\frac{1+t}{2} + (\nu - 1)(t - 2\sqrt{t} + 1) \leq \frac{t^{1-\nu} + t^\nu}{2}$ ($t > 0$). Hence

$$\begin{aligned} \frac{(1 + A^{-\frac{1}{2}}BA^{-\frac{1}{2}})}{2} + (\nu - 1)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} + 1) \\ \leq \frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\nu} + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu}{2}. \end{aligned} \quad (3.3)$$

Multiplying $A^{\frac{1}{2}}$ by the both sides of (3.3) we get

$$A\nabla B + 2(\nu - 1)(A\nabla B - A\sharp_{1/2}B) \leq \frac{A\sharp_{1-\nu}B + A\sharp_\nu B}{2} = H_{1-\nu}(A, B).$$

□

Remark 3.2. Theorem 3.1 also holds for $\nu < \frac{1}{2}$. The case when $\nu < \frac{1}{2}$ is not interesting, since it is less precise than inequality (3.2), but the case of $0 \leq \nu \leq \frac{1}{2}$ coincides with the inequality at [9, Corollary 2].

Applying Theorem 3.1 we get immediately the following result.

Corollary 3.3. *Let $A, B \in \mathbb{B}(\mathcal{H})_{++}$ and $\nu > 1$. Then*

$$H_{1-\nu}(A, B) = A\nabla B$$

if and only if $A = B$.

Applying Lemma 2.1 we get

$$a + a^{-1} \leq a^\nu + a^{-\nu} \quad (a > 0, \nu > 1).$$

Utilizing this inequality, the functional calculus for $A \otimes B^{-1}$ and the definition of the Hadamard product we get the following result.

Proposition 3.4. *Let $A, B \in \mathbb{B}(\mathcal{H})_{++}$ and $\nu > 1$. Then*

- (i) $A \otimes B^{-1} + A^{-1} \otimes B \leq A^\nu \otimes B^{-\nu} + A^{-\nu} \otimes B^\nu$
- (ii) $A \circ B^{-1} + A^{-1} \circ B \leq A^\nu \circ B^{-\nu} + A^{-\nu} \circ B^\nu$.

4. SOME HEINZ TYPE INEQUALITY RELATED TO HADAMARD PRODUCT

In this section, using some ideas of [15] and [16], we show some Heinz type inequalities.

Lemma 4.1. [2, Theorem 1.1.3] *Let $A, B \in \mathcal{P}_n$ and $X \in \mathbb{M}_n$. Then the block matrix*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \text{ is positive semidefinite if and only if } A \geq XB^{-1}X^*.$$

Theorem 4.2. *The two variables function*

$$H(s, t) = A^{1+s} \otimes B^{1-t} + A^{1-s} \otimes B^{1+t}$$

is convex on $[-1, 1] \times [-1, 1]$ and attains its minimum at $(0, 0)$ for all $A, B \in \mathcal{P}_n$.

Proof. Since H is continuous, it is enough to prove

$$H(s_1, t_1) \leq \frac{1}{2}(H(s_1 + s_2, t_1 + t_2) + H(s_1 - s_2, t_1 - t_2))$$

for all $s_1 \pm s_2, t_1 \pm t_2 \in [0, 1]$; see [16]. For $A, B \in \mathcal{P}_n$ and $s_1 \pm s_2, t_1 \pm t_2 \in [0, 1]$ it follows from Lemma 4.1 that the matrices $\begin{pmatrix} A^{1+s_1+s_2} & A^{1+s_1} \\ A^{1+s_1} & A^{1+(s_1-s_2)} \end{pmatrix}, \begin{pmatrix} A^{1-(s_1+s_2)} & A^{1-s_1} \\ A^{1-s_1} & A^{1-(s_1-s_2)} \end{pmatrix},$
 $\begin{pmatrix} B^{1+t_1+t_2} & B^{1+t_1} \\ B^{1+t_1} & B^{1+(t_1-t_2)} \end{pmatrix}$ and $\begin{pmatrix} B^{1-(t_1+t_2)} & B^{1-t_1} \\ B^{1-t_1} & B^{1-(t_1-t_2)} \end{pmatrix}$ are positive semidefinite. Hence the matrices

$$X = \begin{pmatrix} A^{1+s_1+s_2} \otimes B^{1-(t_1+t_2)} + A^{1-(s_1+s_2)} \otimes B^{1+t_1+t_2} & A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} \\ A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} & A^{1+(s_1-s_2)} \otimes B^{1-(t_1-t_2)} + A^{1-(s_1-s_2)} \otimes B^{1+(t_1-t_2)} \end{pmatrix}$$

is positive semidefinite. Similarly,

$$Y = \begin{pmatrix} A^{1+(s_1-s_2)} \otimes B^{1+(t_1-t_2)} + A^{1-(s_1-s_2)} \otimes B^{1-(t_1-t_2)} & A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} \\ A^{1+s_1} \otimes B^{1-t_1} + A^{1-s_1} \otimes B^{1+t_1} & A^{1+s_1+s_2} \otimes B^{1-(t_1+t_2)} + A^{1-(s_1+s_2)} \otimes B^{1+t_1+t_2} \end{pmatrix}$$

is positive semidefinite. Thus

$$X + Y = \begin{pmatrix} H(s_1 + s_2, t_1 + t_2) + H(s_1 - s_2, t_1 - t_2) & 2H(s_1, t_1) \\ 2H(s_1, t_1) & H(s_1 + s_2, t_1 + t_2) + H(s_1 - s_2, t_1 - t_2) \end{pmatrix}$$

is positive semidefinite and therefore

$$\begin{pmatrix} I_n & -I_n \\ 0 & 0 \end{pmatrix} (X + Y) \begin{pmatrix} I_n & 0 \\ -I_n & 0 \end{pmatrix}$$

is positive semidefinite. Hence $H(s_1 + s_2, t_1 + t_2) + H(s_1 - s_2, t_1 - t_2) - 2H(s_1, t_1) \geq 0$, which proves the convexity of H . Further note that $H(s, t) = H(-s, -t)$ $s, t \in [0, 1]$. This together with the convexity of H imply that H attains its minimum at $(0, 0)$. \square

If in Theorem 4.2 we replace s, t, A, B by $2s - 1, 2t - 1, A^{\frac{1}{2}}, B^{\frac{1}{2}}$, respectively, we reach the following result.

Corollary 4.3. *The two variables function*

$$K(s, t) = A^s \circ B^{1-t} + A^{1-s} \circ B^t \quad (A, B \in \mathcal{P}_n)$$

is convex on $[0, 1] \times [0, 1]$ and attains its minimum at $(\frac{1}{2}, \frac{1}{2})$.

Aujla et al. [15] showed that

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^t \circ B^{1-t} + A^{1-t} \circ B^t||| \leq |||A + B|||,$$

where $A, B \in \mathcal{P}_n$ and $t \in [0, 1]$. Now, we are ready to state our last result.

Corollary 4.4. *Let $A, B \in \mathcal{P}_n$ and $s, t \in [0, 1]$. Then*

$$2|||A^{\frac{1}{2}} \circ B^{\frac{1}{2}}||| \leq |||A^s \circ B^{1-t} + A^{1-s} \circ B^t||| \leq \max\{|||(A + B) \circ I|||, |||(A \circ B) + I|||\}.$$

Proof. Let $K(s, t) = A^s \circ B^{1-t} + A^{1-s} \circ B^t$. If we put $G(s, t) = |||K(s, t)|||$, then by the convexity of K and Fan Dominance Theorem [2, p. 58] (see also [14]), the function G is convex on $[0, 1] \times [0, 1]$, and attains minimum at $(\frac{1}{2}, \frac{1}{2})$. Hence we have the first inequality. In addition, since the function G is continuous and convex on $[0, 1] \times [0, 1]$, it follows that G attains its maximum at the vertices of the square. Moreover, due to the symmetry there are only two possibilities for the maximum. \square

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¹ DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159, MASHHAD 91775, IRAN

E-mail address: Mojtaba.Bakherad@yahoo.com; bakherad@member.ams.org

² DEPARTMENT OF PURE MATHEMATICS, CENTER OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159, MASHHAD 91775, IRAN

E-mail address: moslehian@um.ac.ir, moslehian@member.ams.org